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A Method of Construction of Triangular PBIB Designs

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SUMMARY

A new series of 2-associate triangular PBIBD is being presented in this paper.

Key words : T_m- association scheme; PBIBD; BIBD.

1. Introduction

John [1] extended the triangular association scheme to three associate classes. Later on Saha [2] had introduced triangular association scheme, representing each treatment by an m-set and giving the name "m-associate triangular (T_m-) association scheme". Many of PBIBD's based on such association scheme are known. Sinha [3], [4], [5], and [6] had also constructed several series of such designs.

The 2-associate triangular association scheme is defined as below.

Definition 1. Let there be $\binom{n}{2}$ treatments represented by all the possible sets $[\theta_1, \theta_2]$, assuming that $[\theta_1, \theta_2]$ and $[\theta_2, \theta_1]$ represent the same treatment, where the θ's are integers satisfying $0 < \theta_1, \theta_2 \leq n$. If two treatments have an integer in common they are first associate, otherwise second associate.

The parameters of the association scheme are :

$$v = \binom{n}{2}, \quad n_1 = 2(n-2), \quad p_{11}^1 = n-2, \quad p_{11}^2 = 4$$

Notation 1. For a given set S, by $\|S\|$ means the cardinality of S.

2. The Construction

Given a set, $X = \{1, 2, \ldots, n\}$ consider a partition of X into three disjoint sets, S_1, S_2, S_3, $\| S_i \| = s_i$; $i = 1, 2, 3$ and $s_1 \neq s_2$. A block based on S_1 S_2, S_3 is constructed so that its contents are all possible 2-set : one element coming from S_1 and another from S_2 and all possible 2-sets : both elements coming from S_3. Representing all possible 2-sets of X, the treatments of a design and constructing the blocks for all possible partitions of X, a series of design given in the below theorem is obtained.

Theorem 1. A series of two-associate class triangular PBIBD's with the following parameters always exists.

$$v = \binom{n}{2}, \quad b = \binom{n}{s_1}\binom{n-s_1}{s_2}, \quad k = s_1 s_2 + \alpha\binom{s_3}{2}$$

$$r = 2\binom{n-2}{s_1-1}\binom{n-s_1-1}{s_2-1} + \alpha\binom{n-2}{s_1}\binom{n-s_1-2}{s_2}$$

$$\lambda_1 = \delta_1\binom{n-3}{s_1-1}\binom{n-s_1-2}{s_2-2} + \delta_2\binom{n-3}{s_1-2}\binom{n-s_2-1}{s_2-1} + \delta_3\binom{n-3}{s_1}\binom{n-s_1-3}{s_2}$$

$$\lambda_2 = 4\delta_4\binom{n-4}{s_1-2}\binom{n-s_1-2}{s_2-2} + \delta_5\binom{n-4}{s_1}\binom{n-s_1-4}{s_2} + 4\alpha\binom{n-4}{s_1-1}\binom{n-s_1-3}{s_2-1}$$

where

$\alpha = 0$, if $s_3 < 2$, = 1, otherwise, $\quad \delta_1 = 0$, if $s_2 < 2$; = 1 otherwise, $\quad \delta_2 = 0$, if $s_1 < 2$; = 1, otherwise,

$\delta_3 = 0$, if $s_3 < 3$; = 1, otherwise, $\quad \delta_4 = 0$, if one of s_1 and $s_2 < 2$; = 1, otherwise $\quad \delta_5 = 0$, if $s_3 < 4$; = 1, otherwise

Proof: It can be easily seen that v is equal to $\binom{n}{2}$. For the construction of block, there are $\binom{n}{s_1}\binom{n-s_1}{s_2}$ distinct partitions of X, which gives the number of blocks. The block size is equal to the sum of the number of 2-sets : one element coming from S_1 and another from S_2 and number of 2-sets : both elements coming from S_3. That is $s_1\ s_2 + \binom{s_3}{2}$.

A treatment $\{\theta_1, \theta_2\}$ occurs in a block, if the partition of X makes that (i) θ_1 (θ_2) and θ_2 (θ_1) belong to S_2 or (ii) θ_1 and θ_2 belong to S_3. The number of such partitions is

$$2\binom{n-2}{s_1-1}\binom{n-s_1-1}{s_2-1}\binom{n-s_1-s_2}{s_3}+\binom{n-2}{s_1}\binom{n-s_1-2}{s_2}\binom{n-s_1-s_2-2}{s_3-2}$$

which is the replication number of the design.

Any two first associates, $\{\theta_1, \theta_2\}$ and $\{\theta_1, \theta_3\}$ appear together in a block when its partition of X is such that (i) θ_1 (both θ_2 and θ_3) belong to S_1 and both θ_2 and θ_3 (θ_1) belong to S_2 or (ii) θ_1, θ_2, θ_3 belong to S_3.

Any two second associates, $\{\theta_1, \theta_2\}$ and $\{\psi_1, \psi_2\}$ occur together in a block if X is partitioned such that

(i) (θ_1, ψ_1), (θ_2, ψ_2) belong to S_1, S_2 respectively, or

(ii) (θ_1, ψ_2), (θ_2, ψ_1) belong to S_1, S_2 respectively, or

(iii) (θ_2, ψ_1), (θ_1, ψ_2) belong to S_1, S_2 respectively, or

(iv) (θ_2, ψ_2), (θ_1, ψ_1) belong to S_1, S_2 respectively, or

(v) $\theta_1, \theta_2, \psi_1, \psi_2$ belong to S_3, or

(vi) θ_1 (θ_2), θ_2 (θ_1), both ψ_1 and ψ_2 belong to S_1, S_2, S_3 respectively, or

(vii) ψ_1 (ψ_2), ψ_2 (ψ_1), both θ_1 and θ_2 belong to S_1, S_2, S_3 respectively.

Counting straight forward the partitions of X, the value of λ_1 and λ_2 are obtained.

Under the condition s_1=s_2=s (say), each block in the above construction gets repeated twice. Keeping only one copy of each block, we get :

Corollary 1. A series of two-associate class triangular PBIBD with the following parameters always exists.

$$v = \binom{n}{2},\ b = \frac{1}{2}\binom{n}{s}\binom{n-s}{s},\ k = s^2 + \alpha\binom{n-2s}{2}$$

$$r = \binom{n-2}{s-1}\binom{n-s-1}{s-1} + \frac{1}{2}\alpha\binom{n-2}{s}\binom{n-s-2}{s}$$

$$\lambda_1 = \delta_1\binom{n-3}{s-1} + \frac{1}{2}\delta_2\binom{n-3}{s}\binom{n-s-3}{s}$$

$$\lambda_2 = 2\,\delta_1 \binom{n-4}{s-2}\binom{n-s-2}{s-2} + \frac{1}{2}\,\delta_3 \binom{n-4}{s}\binom{n-s-4}{s} + 2\alpha \binom{n-4}{s-1}\binom{n-s-3}{s-1}$$

where

$\delta_1 = 0$, if $s < 2$; $\quad \delta_2 = 0$, if $n < 3 + 2s$

$= 1$, otherwise, $\quad = 1$, otherwise

$\delta_3 = 0$, if $n < 2(s+2)$; $\quad \alpha = 0$, if $n < 2(s+1)$

$= 1$, otherwise, $\quad = 1$, otherwise

Two series of 2-associate triangular PBIBD are given as example of Theorem 1 and Corollary 1 respectively.

Example 1. Letting $s_1 = 2$, $s_2 = 1$, $s_3 = 2$, the parameters of the design are $v = 10$, $b = 30$, $k = 3$, $r = 9$, $\lambda_1 = 1$, $\lambda_2 = 4$.

Example 2. For $s = 2$, $n = 6$, we get a series of 2-associate triangular PBIBD with the parameters $v = 15$, $b = 45$, $k = 5$, $r = 15$, $\lambda_1 = 3$, $\lambda_2 = 6$.

Remark 1. The possible different relations on n and s, under which PBIBD given in Corollary 1, becomes a BIBD are as follows :

If (i) $s \geq 2$, $n \geq 2(s+2)$, when
$(n-3)\,(s-1)!\; q! = (n-s-2)!\; [2s\,(s-1)^2 + q\,(q-1)\,(4s-n+2)]$

(ii) $s \geq 2$, $n < 2(s+2)$,
when $(n-3)\,(s-2)!\, q! = 2\,(n-s-2)!\,(s-1)$

(iii) $s \geq 2$, $n = 2(s+2)$,
when $(n-3)\,(s-1)!\; q! = 2\,(n-s-2) - [(s-1)^2 + q\,(q-1)]$

(iv) $s = 1$, $n \geq 2(s+2)$, the BIBD is with parameters $v = 15 = b$, $k = 7 = r$, $\lambda = 3$

(v) $s \geq 2$, $n = 2s + 3$,
when $2s\,(n-3)\; s!\; q! = (n-s-2)!\; [4s^2\{(s-1)^2 + q\,(q-1)\} - (n-3)\; q\,(q-1)\,(q-2)]$

where $q = n - 2s$

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